MATH 1010E University Mathematics
Lecture Notes (week 1)
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1 Outline of the course

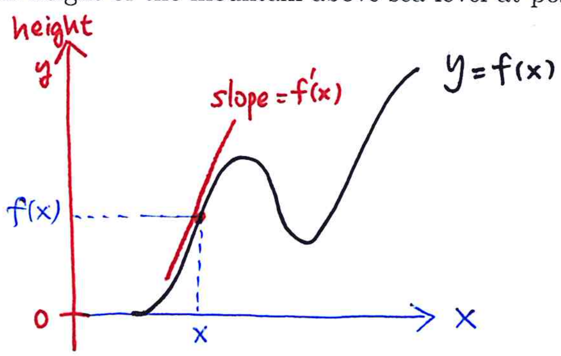
This course is about "single variable calculus". So what is "calculus" all about? It is a useful mathematical tool that helps us better understand something called "functions" (for example, $f(x) = x^2, f(x) = \cos x$ or $f(x) = \log x$). Calculus can be divided into three parts:

(i) differentiation;

(ii) integration; and

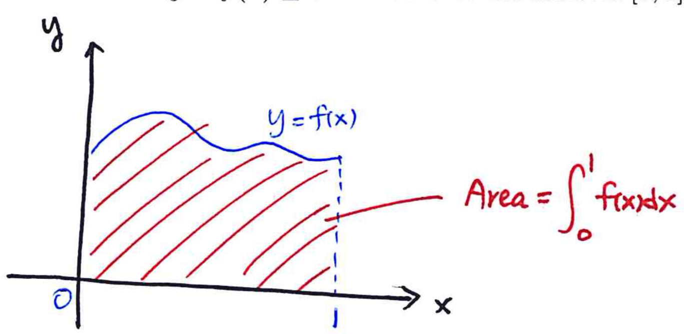
(iii) fundamental theorem of calculus.

"*Differentiation*" is about the rate of change of a function. It is also related to the concept of slope or "steepness". For example, imagine we have a mountain whose shape is represented by the graph of a function $f(x)$, i.e. $f(x)$ is the height of the mountain above sea level at position x:



We want to ask the question how steep the mountain is. First of all, since some parts of the mountain are steeper and some are relatively flat, the answer ought to depend on the position x. In fact, the "derivative of f", denoted by $f'(x)$ is a function of position which equals to the slope of the mountain at position x. Another application comes from physics. In mechanics, a fundamental question is that whether we can determine the trajectory of an object subject to various forces, i.e. we want to know the position of an object changes in time. To describe the rate of change of position, we need the concept of derivative. In fact, Newton was forced to develop calculus in order to write down his famous Newton's laws in mechanics.

"*Integration*", on the other hand, is a "summing-up" process which allows us to find the length of a curve and the area of a planar domain. For example, if we have a function $y = f(x) \geq 0$ defined over the interval $[0, 1]$,



then the "integral" $\int_0^1 f(x)dx$ is a number equal to the area of the region under the curve over the interval $[0, 1]$.

Fundamental theorem of calculus is a theorem which roughly says that the processes of differentiation and integration are inverse to each other. That is, if we differentiate a function and then integrate it, we should get back the original function. The same happens if you integrate first, then differentiate.

Here is a very rough plan of the whole course:

(1) What is a "function"? What are some examples of "elementary functions" and their basic properties?

(2) What is the concept of "limit"? How do we understand the concept of infinity mathematically?

(3) How to differentiate a function? What is the geometric meaning?

(4) How to integrate a function? What does it represent?

(5) How to relate differentiation and integration - fundamental theorem of calculus?

(6) What are some applications of calculus?

2 What is a function?

Definition 2.1 *A function f consists of two sets D and E, called the* domain *and* codomain *respectively, together with a rule of assignment that every element x in D is associated a unique element y in E. We often represent a function by*

$$f : D \to E$$

$$x \mapsto y.$$

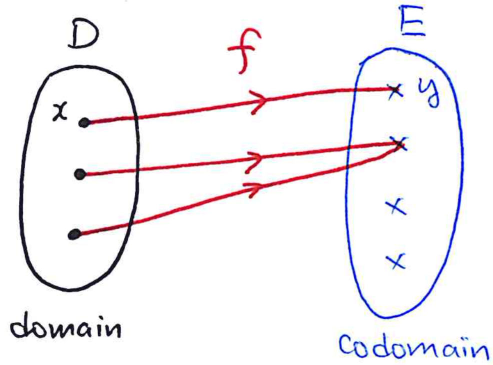
We also write $y = f(x)$ to denote the unique element associated to x, called the image *of x under f.*

For example, the following are examples of functions

(i) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$,

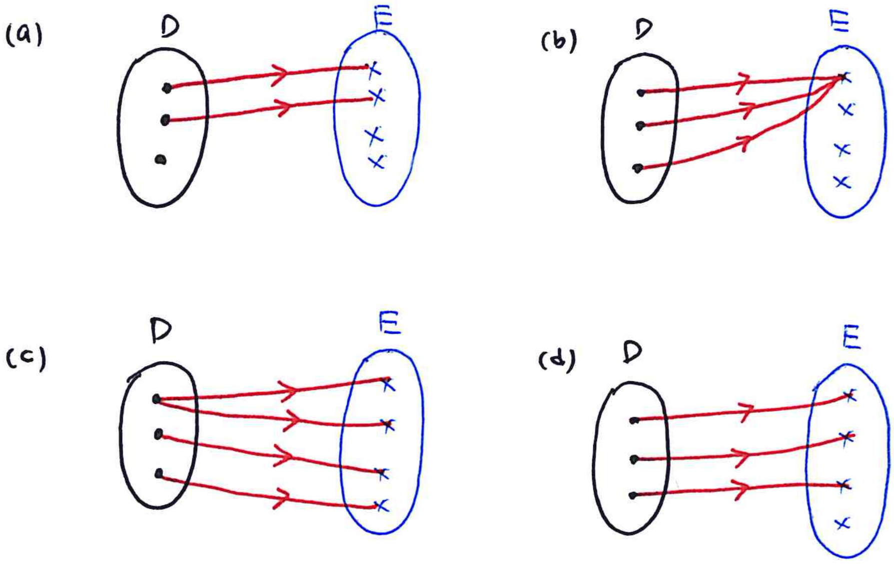
(ii) $f : [0, \infty) \to \mathbb{R}$, $f(x) = x$.

Note that technically speaking, (i) and (ii) are different functions since they have different domains, even though the rule of assignment (i.e. $x \mapsto x$ is the same). We also use a diagram to describe a function as below:



Remark 2.2 *There is indeed some conditions on the rule of assignment in the definition of a function. We require that EVERY element in D is being associated to a UNIQUE element in E.*

Question: Which of the following diagrams represents a function? Why?



3 Some useful notations

We will use the following notations throughout this course. Note that some of these notations are not universal. They may have different meanings in different books.

$\mathbb{R}$	:	the set of all real numbers
$\mathbb{N}$	:	the set of all natural numbers
$\in$	:	belongs to/in
$\subset$	:	is contained in
[a,b]	:	the closed interval $\{x \in \mathbb{R} \mid a \leq x \leq b\}$
(a, b)	:	the open interval $\{x \in \mathbb{R} \mid a < x < b\}$
[a,b)	:	the half open interval $\{x \in \mathbb{R} \mid a \leq x < b\}$
$\exists$	:	there exists/for some
$\exists!$	:	there exists a unique
$\forall$	:	for all/for any
$:=$	:	is defined as
$\Rightarrow$	:	implies
$\Leftrightarrow$	:	is equivalent to
$s.t.$	:	such that

For example, consider the statement "For any real number x greater than zero, there is a unique real number y greater than zero such that the square of y is equal to x.", if we use the notations above, we can write it as

$$\forall x \in (0, \infty),\ \exists! y \in (0, \infty) \text{ s.t. } y^2 = x.$$

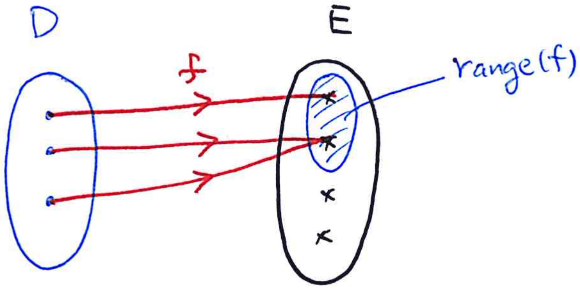
Exercise: Write down the definition of a function using these notations.

4 Range vs codomain

Given a function $f : D \to E$, the *range* of f is defined to be

$$\text{range}(f) = f(D) := \{y \in E \mid \exists x \in D \text{ s.t. } f(x) = y\}.$$

Note that by definition, $f(D) \subset E$. (It could happen that $f(D) = E$, when the function f is "surjective".)



Question: What is the range of the function $f : D \to \mathbb{R}$, $f(x) = x^2$, when

(i) $D = \mathbb{R}$,

(ii) $D = [0, \infty)$,

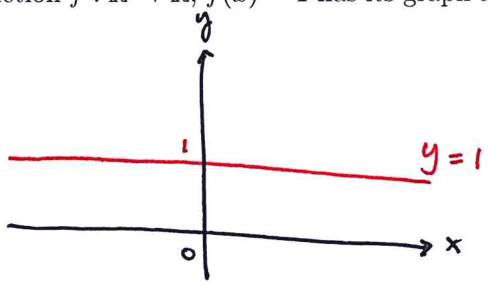
(iii) $D = [1, 2)$.

5 Function as formulae and graphs

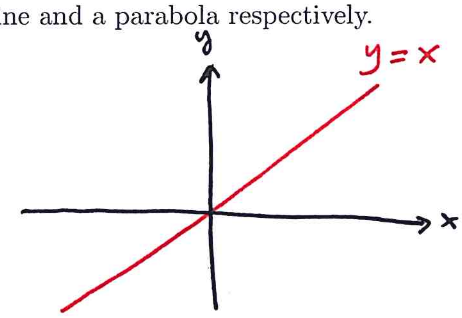
The rule of assignment of a function is often described by a "formula", e.g. the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1$ assigns every $x \in \mathbb{R}$ with the number 1. Another way to describe a function is to draw its graph:

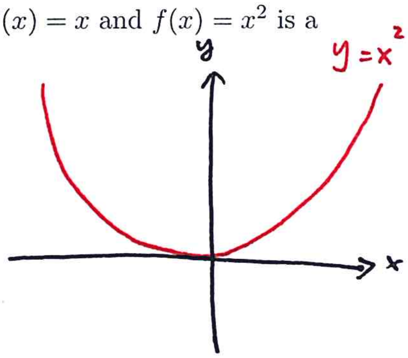
$$\text{graph}(f) := \{(x, f(x)) \mid x \in D\}.$$

For example, the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1$ has its graph as a horizontal line of height 1:

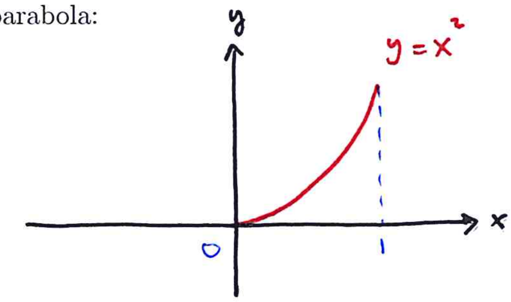


The graph of the functions $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x$ and $f(x) = x^2$ is a line and a parabola respectively.





Note that the domain of a function also affects the graph. For example, the function $f : [0, 1] \to \mathbb{R}$, $f(x) = x^2$ has the following graph which consists of just a piece of the parabola:



Question: Sketch the graph of the following functions: $f(x) = x^3$, $f(x) = x^2 + x - 1$, $f(x) = 1/x$ and $f(x) = \log x$. Where are the functions defined?

Not every curve on the plane comes from the graph of a function. For example, the circle $\{(x, y) \mid x^2 + y^2 = 1\}$ is not the graph of ANY function (why?). Try to come up with a good way to test when is a given curve the graph of a function.

6 Injective and surjective functions

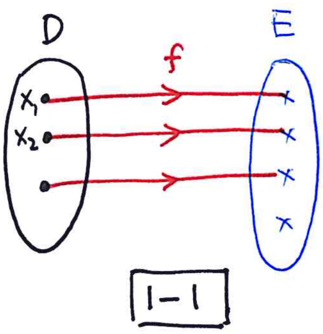
Definition 6.1 *A function $f : D \to E$ is* injective *(or 1-1) if*

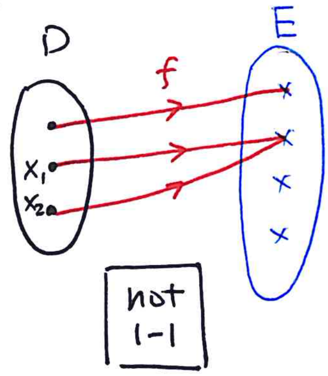
$$f(x_1) = f(x_2) \;\Rightarrow\; x_1 = x_2.$$

or equivalently (in its contrapositive form),

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

In other words, a function is 1-1 if no two elements in the domain are assigned to the same element in the codomain.

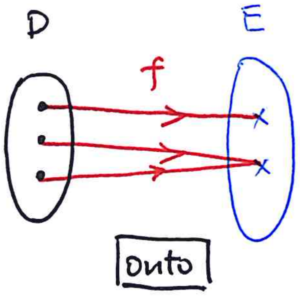


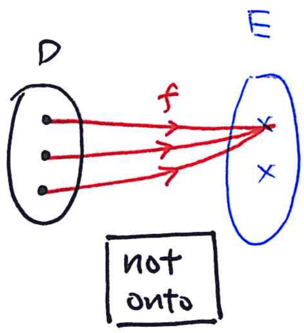


Definition 6.2 *A function $f : D \to E$ is* surjective *(or onto) if*

$$\forall y \in E,\ \exists x \in D \ s.t.\ f(x) = y.$$

In other words, a function is onto if no elements in E is being left over. This is the same as saying that range$(f) = E$.





As the following example shows, whether a function is 1-1 or onto is related to the question of whether the equation

$$y = f(x)$$

can be solved for all y (existence of solution $\Leftrightarrow$ subjectivity), and for those whose solution exists, is it unique (uniqueness of solution $\Leftrightarrow$ injectivity).

Example 6.3 *Show that $f : [0,\infty) \to \mathbb{R}$, $f(x) = x^2$ is 1-1 but not onto.*

Solution: To show that f is 1-1, we need to prove the statement "$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$". Suppose we have ANY $x_1, x_2 \in D$ such that $f(x_1) = f(x_2)$. By definition, it means $x_1^2 = x_2^2$. Taking square root on both sides, we have $x_1 = \pm x_2$. However, since both $D = (0,\infty)$, i.e. $x_1, x_2 > 0$, therefore, we must have $x_1 = x_2$. Since x_1 and x_2 are arbitrary, we have shown that f is 1-1.

To show that f is not onto, we just have to come up with a "counterexample" y in the domain $E = \mathbb{R}$ such that the equation $f(x) = y$ has no solution of $x \in D$. For example, if we take $y = -1$, then $\nexists x \in (0,\infty)$ s.t. $f(x) = x^2 = -1$. Therefore, f is not onto since the element $-1 \in E$ is being left over.

Question: What is the range of f in the above example? What if the domain is $D = \mathbb{R}$? Would the answer change if I change the domain?

Example 6.4 *Consider the function $f : D \to \mathbb{R}$ defined by*

$$f(x) = \frac{10x - 11}{2x^2 - 6x + 4},$$

(i) find the maximum domain of definition (i.e. the largest $D \subset \mathbb{R}$) on which f is defined;

(ii) find the range of f with the maximal domain in (i);

(iii) Is f 1-1? Is f onto? Prove it.

Solution: (i) The function makes sense except when the denominator vanishes, i.e.

$$2x^2 - 6x + 4 = 0,$$

which has only two solutions $x = 1$ and $x = 2$. Therefore, the maximal domain of definition is $D = \mathbb{R} \setminus \{1, 2\}$.

(ii) To find the range of f, we want to see for which $c \in \mathbb{R}$ does the equation $f(x) = c$ have a solution. Therefore, we fix some $c \in \mathbb{R}$ and consider the equation

$$\frac{10x - 11}{2x^2 - 6x + 4} = c.$$

Rearranging the terms we obtain

$$2cx^2 - (6c + 10)x + (4c + 11) = 0,$$

which is a quadratic equation in x (remember c is considered fixed). We want to see for which value of c does the equation have a solution.

Case 1: $c = 0$. The equation reduces to a linear equation

$$-10x + 11 = 0,$$

which has a unique solution $x = 11/10$.

Case 2: $c \neq 0$. The quadratic equation has a solution if and only if the discriminant is nonnegative. Recall that the discriminant Δ of a quadratic equation $ax^2 + bx + c = 0$ is defined to be $\Delta := b^2 - 4ac$ and

- when $\Delta > 0$, there are two distinct real solutions.
- when $\Delta = 0$, there is only one real solution.
- when $\Delta < 0$, there is no real solution.

Therefore, we have the following condition on c which is necessary for the existence of a solution x,

$$(6c + 10)^2 - 8c(4c + 11) \geq 0,$$

which simplifies to

$$c^2 + 8c + 25 \geq 0.$$

However, since $c^2 + 8c + 25 = (c+4)^2 + 9 > 0$ for ANY $c \in \mathbb{R}$. Therefore, the discriminant $\Delta > 0$ for any $c \neq 0$ and hence there are two distinct solutions x_1 and x_2 for any given $0 \neq c \in \mathbb{R}$. This implies that range$(f) = \mathbb{R}$.

(iii) From (ii), since range$(f) = \mathbb{R}$, we have f is onto. From the proof of (ii), since there are two distinct solutions $x_1 \neq x_2$ such that $f(x_1) = f(x_2) = c$ whenever $c \neq 0$, therefore f is not 1-1.

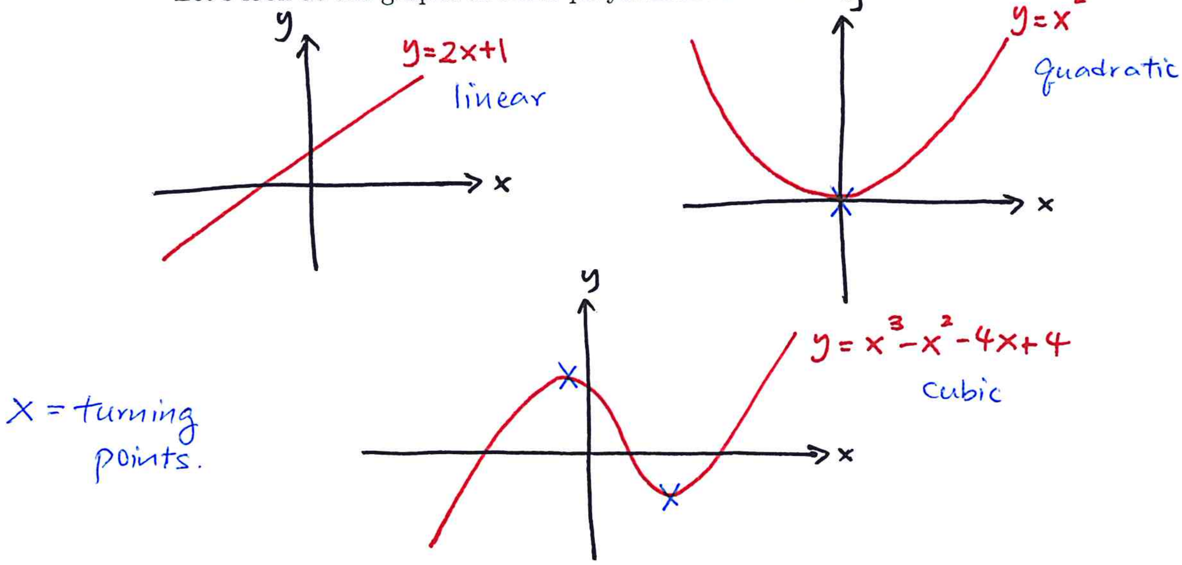
7 Polynomials

Definition 7.1 *A* polynomial of degree n *is a function* $f : \mathbb{R} \to \mathbb{R}$ *of the form*

$$f(x) := \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ *are called the* coefficients *of the polynomial and* $a_n \neq 0$.

Let's look at the graphs of some polynomial functions.



Question: "Prove" that a polynomial of degree n can have at most $n-1$ "turning points".

8 Exponential and logarithm

Definition 8.1 *The* exponential function *is defined to be the function*

$$exp : \mathbb{R} \to \mathbb{R}$$

$$exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots .$$

Note that the exponential function is defined by an infinite series, i.e.a polynomial of infinite degree. The meaning is that when a number x is fixed, we substitute the number into the place of x in the series and then calculate the infinite sum. For example,

$$\exp(0) = 1 + 0 + 0 + 0 + \cdots = 1.$$

When $x \neq 0$, it is more complicated. For example, when $x = 1$,

$$\exp(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots .$$

It is not so clear what is the numerical value of the infinite sum on the right hand side. In fact, when dealing with infinite sum, one has to be very careful whether things make sense. For example, one could easily write down an infinite sum that does not sum up to a finite number:

$$1+1+1+1+1+\cdots \text{``} = +\infty\text{''?},$$

or sums that are ambiguous:

$$1-1+1-1+1-1+\cdots,$$

since we get different answers from different calculations:

$$1+(-1+1)+(-1+1)+\cdots = 1+0+0+\cdots = 1,$$

$$(1-1)+(1-1)+(1-1)+\cdots = 0+0+0+\cdots = 0.$$

We are not going to discuss in detail about when an infinite sum behaves well. We just assume the fact that the infinite sum in the definition of $\exp(x)$ makes sense for any $x \in \mathbb{R}$.

One may ask why we define a function in such a funny way. One answer is that we know how to manipulate polynomials very well, and that a "good" infinite polynomial behaves in many ways similar to a polynomial. For example, one can prove the following theorem.

Theorem 8.2 *For any $x, y \in \mathbb{R}$, we have*

$$exp(x+y) = exp(x)exp(y).$$

Proof: We will give a sketch of the proof and leave the rigorous mathematical proof to you as an exercise.

From the definition of exp, we have

$$\exp(x)\exp(y) = \left(1+x+\frac{x^2}{2}+\cdots\right)\left(1+y+\frac{y^2}{2}+\cdots\right).$$

Expanding the right hand side, we get

$$\left(1+x+\frac{x^2}{2}+\cdots\right)+\left(y+xy+\frac{x^2y}{2}+\cdots\right)+\left(\frac{y^2}{2}+\frac{xy^2}{2}+\frac{x^2y^2}{4}+\cdots\right).$$

Collecting terms of the same degree, we have

$$1+(x+y)+\frac{(x+y)^2}{2}+(\text{terms of higher degree}),$$

whose first few terms agree with $\exp(x+y)$.

Question: Give a rigorous proof that all the terms agree using the binomial formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient.

Definition 8.3 *The logarithm function ln is defined by*

$$ln(1+x) := \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

for any $x \in (-1,1]$.

Remark 8.4 *Note that the series in the definition of $ln(1+x)$ only makes sense for $x \in (-1,1]$. However, we can actually extend the definition to all $x \in (-1,\infty)$ so that the theorem below holds. From now on, we assume $ln(1+x)$ is defined on $(-1,\infty)$ and whenever we use the series definition, we restrict to $x \in (-1,1]$.*

Theorem 8.5 *The following identities are true:*

(i) $exp(ln(y)) = y$ for any $y \in (0,\infty)$,

(ii) $ln(exp(x)) = x$ for any $x \in \mathbb{R}$.

Question: Prove the theorem above. You can follow Exercise 1.11 in the textbook.

9 Inverse of a function

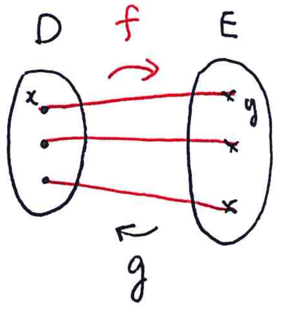
The theorem above says that the exponential and logarithm functions are inverse to each other.

Definition 9.1 *We say that a function $g : E \to D$ is the* inverse *of the function $f : D \to E$ if*

(i) $g(f(x)) = x$ for all $x \in D$, and

(ii) $f(g(y)) = y$ for all $y \in E$.

We often denote $g = f^{-1}$.



Question: Show that if f has an inverse, then f must be 1-1 and onto. Prove that the converse also holds: if f is 1-1 and onto, then the inverse f^{-1} exists.